

# Hitting all longest cycles in a graph

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**Abstract.** Let  $\text{lct}(G)$  be the minimum size of a set of vertices that contains at least one vertex in every longest cycle of a graph  $G$ . We show that  $\text{lct}(G) = 1$  if  $G$  is a 3-tree, and that  $\text{lct}(G) \leq 2$  if  $G$  is a 2-connected partial 3-tree.

## 1. Introduction

It is known that, in every 2-connected graph, every pair of longest cycles intersect each other in at least two vertices. A natural question asks whether all longest cycles have a vertex in common in 2-connected graphs. (If the graph is not 2-connected, two longest cycles can be disjoint.) This has in general a negative answer, as the Petersen's graph shows. However, there are some graph classes for which this question has a positive answer, such as classes containing only Hamiltonian graphs [Thomas and Yu 1994, Tutte 1956], and dually chordal graphs [Jobson et al. 2016]. In this paper we show that the class of 3-trees also has a positive answer to this question. Observe that 3-trees are not dually chordal graphs, as they are not clique-Helly graphs, so they are not included in the class addressed by Jobson *et al.* [Jobson et al. 2016].

When one cannot find a vertex that is common to all longest cycles in a graph, it is interesting to look for a set of vertices such that every longest cycle has at least one vertex in that set. Such a set is called a longest cycle transversal, or just a transversal, and we look for small transversals, possibly of minimum size. The minimum size of a transversal is denoted by  $\text{lct}(G)$ . When we cannot determine  $\text{lct}(G)$  exactly, it is interesting to search for a good upper bound for it. For every 2-connected graph  $G$  with  $n$  vertices,  $\text{lct}(G) \leq \lceil n/3 \rceil$  [Thomassen 1978]. This bound was later improved to  $\text{lct}(G) \leq \lceil \frac{n}{3} - \frac{n^{2/3}}{36} \rceil$  [Rautenbach and Sereni 2014]. Restating what was written at the end of the previous paragraph, here we show that  $\text{lct}(G) = 1$  when  $G$  is a 3-tree. We also show that, when  $G$  is a 2-connected partial 3-tree,  $\text{lct}(G) \leq 2$ .

## 2. Preliminaries

Let  $C$  be a cycle in a graph  $G$ . Sometimes we will refer to  $C$  as the set of vertices of  $C$ . We denote by  $|C|$  the length of  $C$ , that is, the number of edges in  $C$ . A triangle in  $G$  is a cycle of length three. Let  $S$  be a set of vertices of  $G$ . Given a cycle  $C$  with at least one vertex not in  $S$ , we say that  $S$  fences  $C$  if  $C - S$  is contained in a single connected component of  $G - S$ , otherwise we say that  $C$  crosses  $S$ . For an integer  $t$  and a set  $S$  of vertices, we say that a cycle  $C$   $t$ -touches  $S$  if  $C$  intersects  $S$  at exactly  $t$  vertices. Note that if  $C$  1-touches  $S$  then  $C$  is fenced by  $S$ .

**Lemma 1.** Let  $G$  be a 2-connected graph with a triangle  $\Delta$ . Let  $\mathcal{C}$  be the set of all longest cycles in  $G$  that cross  $\Delta$ . At least two vertices of  $\Delta$  are in all cycles of  $\mathcal{C}$ .

Let  $X$  and  $Y$  be two subsets of  $V(G)$ . We denote by  $\text{Comp}_Y(X)$  the union of the vertex set of the connected components of  $G - Y$  containing at least one vertex of  $X \setminus Y$ . If  $C$  is a cycle, then we may write  $\text{Comp}_Y(C)$  instead of  $\text{Comp}_Y(V(C))$ . Also, if  $v$  is a vertex, we may write  $\text{Comp}_Y(v)$  instead of  $\text{Comp}_Y(\{v\})$ . The next lemma is an adaptation of Lemma 6 of de Rezende *et al.* [De Rezende et al. 2013].

**Lemma 2.** *Let  $G$  be a 2-connected graph with a triangle  $\Delta$ . If there are longest cycles  $C'$  and  $C''$  such that  $\text{Comp}_\Delta(C') \cap \text{Comp}_\Delta(C'') = \emptyset$ , then one of the vertices of  $\Delta$  is in all longest cycles of  $G$ .*

A 3-tree is defined recursively as follows. A graph that is a triangle is a 3-tree. Any graph obtained from a 3-tree by adding a new vertex and making it adjacent to all the vertices of an existing triangle is also a 3-tree. We say that a graph that is a triangle is a trivial 3-tree, while all the other 3-trees are nontrivial. A graph is a partial 3-tree if it is a subgraph of a 3-tree. Forbidden minors are known for partial 3-trees.

**Lemma 3.** [Arnborg *et al.* 1990] *There is no  $K_5$ -minor in a partial 3-tree.*

### 3. 3-trees

Our goal is to show that all longest cycles intersect in 3-trees. Next we will generalize Lemmas 7 and 8 of de Rezende *et al.* [De Rezende et al. 2013] for 3-trees.

**Lemma 4.** *If  $K$  is a  $K_4$  in a graph  $G$ , then either  $G$  contains a  $K_5$ -minor or*

$$\cap\{\text{Comp}_\Delta(v_\Delta) : \Delta \text{ is a triangle in } K\} = \emptyset,$$

where  $v_\Delta$  is the vertex of  $K$  not in  $\Delta$ .

Let  $K$  be a  $K_4$  in a connected graph  $G$ . We say that  $K$  is 4-fencing if, for each triangle  $\Delta$  in  $K$ , there is a longest cycle  $C$  in  $G$  such that  $C$  is fenced by  $\Delta$ , intersects  $\Delta$  at most twice, and satisfies  $\text{Comp}_\Delta(C) = \text{Comp}_\Delta(v_\Delta)$ , where  $v_\Delta$  is the vertex of  $K$  not in  $\Delta$ .

**Lemma 5.** *If, for every triangle  $\Delta$  in a nontrivial 3-tree  $G$ , there is a longest cycle fenced by  $\Delta$  and intersecting  $\Delta$  at most twice, then  $G$  contains a 4-fencing  $K_4$ .*

The next lemma is basically a corollary of Lemma 2.

**Lemma 6.** *Let  $G$  be a 2-connected graph. If  $K$  is a 4-fencing  $K_4$  in  $G$  and  $C$  is a longest cycle such that  $\text{Comp}_K(C) \cap \text{Comp}_\Delta(v_\Delta) = \emptyset$ , where  $\Delta$  is a triangle in  $K$  and  $v_\Delta$  is the vertex in  $K$  not in  $\Delta$ , then there is a vertex in  $K$  intersecting all longest cycles in  $G$ .*

**Corollary 7.** *Let  $G$  be a 3-tree. Either there exists a vertex intersecting all longest cycles in  $G$  or every 4-fencing  $K_4$  in  $G$  fences no longest cycle.*

The corollary above implies that, in a 3-tree, either there is a vertex intersecting all longest cycles or every 4-fencing  $K_4$  intersects at least twice every longest cycle, because each such cycle crosses such  $K_4$ . Next lemma strengthens this result.

**Lemma 8.** *Let  $G$  be a 3-tree. Either there exists a vertex intersecting all longest cycles in  $G$ , or no longest cycle in  $G$  2-touches a 4-fencing  $K_4$  in  $G$ .*

So, in a 3-tree  $G$ , either there exists a vertex intersecting all longest cycles in  $G$ , or every longest cycle intersects at least three times each 4-fencing  $K_4$ . If  $G$  has a 4-fencing  $K_4$  and no vertex intersects all longest cycles in  $G$ , then, for every triangle  $\Delta$  in this  $K_4$ , there must be a longest cycle 3-touching the  $K_4$  at the vertices of  $\Delta$ .

**Theorem 9.** *In every 3-tree  $G$ , there exists a vertex intersecting all longest cycles in  $G$ , that is,  $\text{lct}(G) = 1$ .*

#### 4. Partial 3-trees

Before proving our main result, we give some basic definitions and notation.

A tree-decomposition [Diestel 2010, p. 337] of a graph  $G$  is a pair  $(T, \mathcal{V})$  consisting of a tree  $T$  and a collection  $\mathcal{V} = \{V_t : t \in V(T)\}$  of bags  $V_t \subseteq V(G)$ , that satisfies the following three conditions:

- (T1)  $\bigcup_{t \in V(T)} V_t = V(G)$ ;
- (T2) for every  $uv \in E(G)$ , there exists a bag  $V_t$  such that  $u, v \in V_t$ ;
- (T3) if a vertex  $v$  is in two different bags  $V_{t_1}$  and  $V_{t_2}$ , then  $v$  is also in every bag  $V_t$  such that  $t$  is on the (unique) path from  $t_1$  to  $t_2$  in  $T$ .

The width of  $(T, \mathcal{V})$  is the number  $\max\{|V_t| - 1 : t \in V(T)\}$ , and the tree-width  $\text{tw}(G)$  of  $G$  is the minimum width of any tree-decomposition of  $G$ .

It is known that partial 3-trees are exactly the graphs with tree-width three [Brandstädt et al. 1999]. Sometimes we are interested in particular tree-decompositions. The following proposition is a lemma of Bodlaender.

**Proposition 10.** [Bodlaender 1998] *If  $k$  is the tree-width of a graph  $G$ , then  $G$  has a tree-decomposition  $(T, \mathcal{V})$  of width  $k$  such that  $|V_t| = k + 1$  for every  $t \in V(T)$ , and  $|V_t \cap V_{t'}| = k$  for every  $tt' \in E(T)$ .*

Given a node  $t$  of  $T$ , we say that the connected components of  $T - t$  are the branches of  $T$  at  $t$  [Heinz 2013]. For a node  $t' \in V(T - t)$ , we denote by  $\text{Branch}_t(t')$  the branch of  $T$  at  $t$  where  $t'$  lies. Similarly, for a vertex  $v \in V(G)$ , we denote by  $\text{Branch}_t(v)$  the branch  $\text{Branch}_t(q)$  of  $T$  at  $t$  where  $q$  is a node of  $T$  such that  $v \in V_q$ .

**Lemma 11.** *Let  $G$  be a 2-connected partial 3-tree. Let  $(T, \mathcal{V})$  be a tree-decomposition of  $G$  as described in Proposition 10, and let  $t$  be a node of  $T$ . If  $\text{lct}(G) > 2$ , then there exists a longest cycle in  $G$  that touches  $V_t$  at most once.*

**Theorem 12.** *For every 2-connected partial 3-tree  $G$ ,  $\text{lct}(G) \leq 2$ .*

*Proof.* Let  $(T, \mathcal{V})$  be a tree-decomposition of  $G$  as described in Proposition 10. Assume by contradiction that  $\text{lct}(G) > 2$ . By Lemma 11, for every node  $t \in V(T)$ , there exists a longest cycle that touches  $V_t$  at most once. We create an auxiliary digraph  $D$ , that admits anti parallel arcs, as follows. The nodes of  $D$  are exactly the nodes of  $T$ . Given a node  $t \in V(T)$ , we add the arc  $tt'$  in  $D$  if there is a longest cycle  $C$  fenced by  $V_t$  such that  $\text{Branch}_t(t') = \text{Branch}_t(C)$ . Then, every node of  $D$  is the tail of some arc in  $D$ .

Let  $tt'$  be the last arc of a maximal directed path in  $D$ . As  $T$  is a tree,  $tt'$  is also an arc, which implies that there exist two longest cycles  $C_t$  and  $C_{t'}$  such that  $\text{Branch}_t(C_t) = \text{Branch}_t(t')$  and  $\text{Branch}_{t'}(C_{t'}) = \text{Branch}_{t'}(t)$ , where  $C_t$  touches  $V_t$  at most once, and  $C_{t'}$  touches  $V_{t'}$  at most once. Note that the bags where the vertices of  $C_t$  lie are only in  $\text{Branch}_t(t') \cup \{t\}$ , and that the bags where the vertices of  $C_{t'}$  lie are only in  $\text{Branch}_{t'}(t) \cup \{t'\}$ . As  $\text{Branch}_t(t')$  and  $\text{Branch}_{t'}(t)$  are disjoint,  $C_t \cap C_{t'} \subseteq V_t \cup V_{t'}$ . Note that  $C_t$  does not contain the only vertex in  $V_t \setminus V_{t'}$ , and that  $C_{t'}$  does not contain the only vertex in  $V_{t'} \setminus V_t$ . But then  $C_t$  and  $C_{t'}$  intersect each other in just one vertex, a contradiction to the fact that  $G$  is 2-connected.  $\square$

## 5. Final remarks

Similar proofs can lead to more general results. Namely, that  $\text{lct}(G) \leq \omega(G) - 3$  for every 2-connected chordal graph, where  $\omega(G)$  is the size of a maximum clique of  $G$ , and that  $\text{lct}(G) \leq k - 1$  for every 2-connected graph of tree-width  $k$ . This would imply that  $\text{lct}(G) = 1$  for 2-connected partial 2-trees and for 2-connected planar graphs that are also chordal.

Similar but weaker results can be obtained when considering paths instead of cycles. One difficulty that arises is that paths can have their ends in different connected components. That is, given a longest path  $P$  with ends  $x$  and  $y$ , and a triangle  $\Delta$ , we can have that  $\text{Comp}_{\Delta}(x) \neq \text{Comp}_{\Delta}(y)$ . This generates more possibilities, and for this reason the results are not as strong as for cycles.

## References

- Arnborg, S., Proskurowski, A., and Corneil, D. (1990). Forbidden minors characterization of partial 3-trees. Discrete Mathematics, 80(1):1–19.
- Bodlaender, H. (1998). A partial  $k$ -arboretum of graphs with bounded treewidth. Theoretical Computer Science, 209(1):1–45.
- Brandstädt, A., Le, V., and Spinrad, J. (1999). Graph Classes: A Survey. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA.
- De Rezende, S., Fernandes, C., Martin, D., and Wakabayashi, Y. (2013). Intersecting longest paths. Discrete Mathematics, 313:1401–1408.
- Diestel, R. (2010). Graph Theory, volume 173 of Graduate texts in mathematics. Springer, 4th edition.
- Heinz, M. (2013). Tree-decomposition: Graph minor theory and algorithmic implications. Master’s thesis, Technischen Universität Wien.
- Jobson, A., Kézdy, A., Lehel, J., and White, S. (2016). Detour trees. Discrete Applied Mathematics, 206:73–80.
- Rautenbach, D. and Sereni, J.-S. (2014). Transversals of longest paths and cycles. SIAM Journal of Discrete Mathematics, 28(1):335–341.
- Thomas, R. and Yu, X. (1994). 4-Connected projective-planar graphs are Hamiltonian. Journal of Combinatorial Theory, Series B, 62(1):114–132.
- Thomassen, C. (1978). Hypohamiltonian graphs and digraphs. In Alavi, Y. and Lick, D. R., editors, Proceedings of the International Conference of the Theory and Applications of Graphs (1976), volume 642 of Lecture Notes in Mathematics, pages 557–571. Springer.
- Tutte, W. (1956). A theorem on planar graphs. Transactions of the American Mathematical Society, 82:99–116.