On a joint technique for Hajós’ and Gallai’s Conjectures *

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Abstract. A path (resp. cycle) decomposition of a graph $G$ is a set of edge-disjoint paths (resp. cycles) of $G$ that covers the edge-set of $G$. Gallai (1966) conjectured that every graph on $n$ vertices admits a path decomposition of size at most $\lceil (n + 1)/2 \rceil$, and Hajós (1968) conjectured that every Eulerian graph on $n$ vertices admits a cycle decomposition of size at most $\lceil (n - 1)/2 \rceil$. In this paper, we verify Gallai’s Conjecture for series–parallel graphs, and for graphs with maximum degree 4. Moreover, we show that the only graphs in these classes that do not admit a path decomposition of size at most $\lfloor n/2 \rfloor$ are isomorphic to $K_3$, $K_5$ or $K_5 - e$. The technique developed here is further used to present a new proof of a result of Granville and Moisiadis (1987) that states that Eulerian graphs with maximum degree 4 satisfy Hajós’ Conjecture.

Resumo. Uma decomposição de um grafo $G$ em caminhos (resp. circuitos) é um conjunto de caminhos (resp. circuitos) arestas-disjuntos de $G$ que cobre o conjunto de arestas de $G$. Gallai (1966) conjecturou que todo grafo com $n$ vértices admite uma decomposição em caminhos $D$ tal que $|D| \leq \lceil (n + 1)/2 \rceil$, e Hajós (1968) conjecturou que todo grafo Euleriano com $n$ vértices admite uma decomposição em circuitos $D$ tal que $|D| \leq \lceil (n - 1)/2 \rceil$. Neste trabalho, nós provamos a Conjectura de Gallai para grafos série-paralelos, e para grafos com grau máximo 4. Além disso, nós mostramos que os únicos grafos nessas classes que não admitem uma decomposição $D$ tal que $|D| \leq \lfloor n/2 \rfloor$ são isomorfos a $K_3$, $K_5$ e $K_5 - e$. A técnica desenvolvida aqui é também usada para apresentar uma nova prova de um resultado de Granville e Moisiadis (1987) que diz que grafos Eulerianos com grau máximo 4 satisfazem a Conjectura de Hajós.

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1. Introduction

A decomposition \( D \) of a graph \( G \) is a set \( \{H_1, \ldots, H_k\} \) of edge-disjoint subgraphs of \( G \) that cover the edge-set of \( G \). We say that \( D \) is a path (resp. cycle) decomposition if \( H_i \) is a path (resp. cycle) for \( i = 1, \ldots, k \). We say that a path (resp. cycle) decomposition \( D \) of a graph (resp. an Eulerian graph) \( G \) is minimum if for any path (resp. cycle) decomposition \( D' \) of \( G \) we have \( |D| \leq |D'| \). The size of a minimum path (resp. cycle) decomposition is called the path (resp. cycle) number of \( G \), and is denoted by \( pn(G) \) (resp. \( cn(G) \)). In this paper, we focus in the following conjectures concerning minimum path and cycles decompositions of graphs (see [Bondy 2014, Lovász 1968]).

**Conjecture 1 (Gallai, 1966)** If \( G \) is a connected graph with \( n \) vertices, then \( pn(G) \leq \lfloor \frac{n+1}{2} \rfloor \).

**Conjecture 2 (Hajós, 1968)** If \( G \) is an Eulerian graph with \( n \) vertices, then \( cn(G) \leq \lfloor \frac{n-1}{2} \rfloor \).

Although these conjectures are very similar, the results obtained towards their verification are distinct. In 1968, Lovász proved that a graph with \( n \) vertices can be decomposed into at most \( \lceil n/2 \rceil \) paths and cycles. A consequence of this result is that if \( G \) is a graph with at most one vertex of even degree, then \( pn(G) = \lfloor n/2 \rfloor \). Pyber (1996) and Fan (2005) extended this result, but the conjecture is still open. In [Botler and Jiménez 2017], one of the authors verified Conjecture 1 for a family of even regular graphs, and Jiménez and Wakabayashi (2014) verified it for a family of triangle-free graphs.

In another direction, Geng, Fang and Li (2015) verified Conjecture 1 for maximal outerplanar graphs and 2-connected outerplanar graphs, and Favaron and Kouider (1988) verified it for Eulerian graphs with maximum degree 4. While we were writing this paper, we learned that Bonamy and Perrett [Bonamy and Perrett 2016] verified Conjecture 1 for graphs with maximum degree 5.

Conjecture 2, on the other hand, was only verified for graphs with maximum degree 4 [Granville and Moisiadis 1987] and for planar graphs [Seyffarth 1992].

In this paper, we present a technique that showed to be useful to deal with both Gallai’s and Hajós’ Conjectures. Our technique consists of finding, given a graph \( G \), a special subgraph \( H \), which we call a reducing subgraph of \( G \), that have small path or cycle number compared to the number of vertices of \( G \) that are isolated in \( G - E(H) \).

In this paper we focus on series–parallel graphs and graphs with maximum degree 4. We verify Gallai’s and Hajós’ Conjectures for these classes in Section 2 and 3, respectively. Due to space limitations, we present only the sketch of some proofs.

2. Reducing subgraphs and Gallai’s Conjecture

Let \( G \) be a graph and let \( H \) be a subgraph of \( G \). Given a positive integer \( r \), we say that \( H \) is an \( r \)-reducing subgraph of \( G \) if \( G - E(H) \) has at least \( 2r \) isolated vertices and \( pn(H) \leq r \). The following lemma arises naturally.

**Lemma 1** Let \( G \) be a graph and \( H \subseteq G \) be an \( r \)-reducing subgraph of \( G \). If \( pn(G - E(H)) \leq \lfloor n/2 \rfloor - r \), then \( pn(G) \leq \lfloor n/2 \rfloor \).

In order to verify Conjecture 1 for graphs with maximum degree 4, we first extend the results in [Geng et al. 2015] by proving that Gallai’s Conjecture holds for series–parallel graphs, which are precisely the graphs with no subdivision of \( K_4 \). The proof of the next theorem relies on the fact that series–parallel graphs with at least four vertices
contain at least two non-adjacent vertices of degree at most 2. This fact is easy to verify, since series-parallel graphs are also the graphs with treewidth at most 2.

**Theorem 2** Let \( G \) be a connected graph on \( n \) vertices. If \( G \) has no subdivision of \( K_4 \), then \( pn(G) \leq \lfloor n/2 \rfloor \) or \( G \) is isomorphic to \( K_3 \).

**Sketch of the proof.** For a contradiction, let \( G \) be a minimum counter-example for the statement. It is not hard to verify that \( G \) has at least five vertices. Thus, let \( u, v \) be two non-adjacent vertices of degree at most 2. We can show that \( u \) and \( v \) have at most one neighbor in common, which implies that there is a path \( P \) containing both \( u \) and \( v \) as internal vertices. Let \( H \) be the graph consisting of \( P \) together with the components of \( G - E(P) \) that isomorphic to \( K_3 \). We can show that \( H \) is an \( r \)-reducing subgraph and that \( pn(G - E(H)) \leq \lfloor n/2 \rfloor - r \). Therefore, Lemma 1 concludes the proof.

The same technique verifies Conjecture 1 for planar graphs with girth at least 6.

**Theorem 3** If \( G \) is a planar graph on \( n \) vertices and girth at least 6, then \( pn(G) \leq \lfloor n/2 \rfloor \).

The next theorem verifies Conjecture 1 for graphs with maximum degree 4.

**Theorem 4** If \( G \) is a connected graph on \( n \) vertices and has maximum degree 4, then \( pn(G) \leq \lfloor n/2 \rfloor \) or \( G \) is isomorphic to \( K_3, K_5 \) or to \( K_5^+ \).

**Sketch of the proof.** For a contradiction, let \( G \) be minimum counter-example for the statement. By Theorem 2, we may suppose that \( G \) contains a subdivision \( H \) of \( K_4 \). Let \( v_1, v_2, v_3, v_4 \) be the vertices of \( H \) with degree 3, and let \( S \) be the set of edges incident to \( v_i \) in \( G - E(H) \), for \( i = 1, 2, 3, 4 \). The rest of the proof depends on the structure of the subgraph of \( G \) induced by \( S \). We analyze one of the possible cases. Suppose that there are distinct vertices \( x,y \) in \( V(G) \) such that \( S \subseteq \{ xv_1, xv_2, yv_3, yv_4 \} \). It is not hard to check that \( H + S \) can be decomposed into two paths, and \( v_1, v_2, v_3, v_4 \) are isolated vertices in \( G - E(H) - S \). Now, let \( H' \) be the graph consisting of \( H + S \) together with the components of \( G - E(H) - S \) that are isomorphic to \( K_3, K_5 \), or \( K_5 - e \). Again, we can show that \( H' \) is an \( r \)-reducing subgraph and that \( pn(G - E(H) - S) \leq \lfloor n/2 \rfloor - r \). Lemma 1 concludes the proof.

### 3. Reducing subgraphs and Hajós’ Conjecture

When dealing with Conjecture 2, the same strategy holds: we first verify Conjecture 2 for graphs with no subdivision of \( K_4 \), and then we show how to extend subdivisions of \( K_4 \) in order to obtain a (cycle) reducing subgraph. Given a positive integer \( r \), we say that an Eulerian subgraph \( H \) of an Eulerian graph \( G \) is an r-cycle reducing subgraph of \( G \) if \( G - E(H) \) has at least 2\( r \) isolated vertices and \( cn(H) \leq r \). Analogously to Section 2 we obtain the following Lemma.

**Lemma 5** Let \( G \) be an Eulerian graph and \( H \subset G \) be an r-cycle reducing subgraph of \( G \). If \( cn(G - E(H)) \leq \lfloor (n - 1)/2 \rfloor - r \), then \( cn(G) \leq \lfloor (n - 1)/2 \rfloor \).

The next theorems are the main results of this section.

**Theorem 6** If \( G \) is an Eulerian graph with \( n \) non-isolated vertices and with no subdivision of \( K_4 \), then \( cn(G) \leq \lfloor (n - 1)/2 \rfloor \).

**Sketch of the proof.** For a contradiction, let \( G \) be minimum counter-example for the statement. Let \( u, v \) be vertices of degree at most 2 in \( G \). It is not hard to prove that \( G \)
is 2-connected, hence there is a cycle $C$ in $G$ containing $u$ and $v$. The cycle $C$ is a 1-cycle reducing subgraph of $G$, and by the minimality of $G$, we have $cn(G - E(C)) \leq \lfloor (n - 1)/2 \rfloor - 1$. Therefore, Lemma 5 concludes the proof.

**Theorem 7** If $G$ is an Eulerian graph with $n$ vertices and maximum degree 4, then $cn(G) \leq \lfloor (n - 1)/2 \rfloor$.

**Sketch of the proof.** For a contradiction, let $G$ be minimum counter-example for the statement. By Theorem 6, we may suppose that $G$ contains a subdivision $H$ of $K_4$. Thus, $G - E(H)$ contains four vertices, say $v_1, v_2, v_3, v_4$, with degree 1. We can suppose, without loss of generality, that $G - E(H)$ contains paths $P, Q$ joining $v_1$ to $v_2$ and $v_3$ to $v_4$, respectively. We can prove that the subgraph $H' = H + P + Q$ is an $r$-cycle reducing subgraph of $G$ and that $cn(G - E(H')) \leq \lfloor n/2 \rfloor - r$. Lemma 5 concludes the proof.

4. Concluding remarks

Reducing subgraphs have allowed us to obtain both new results and new proofs for known results. Also, this work provides literature with a technique that can be applied at the same time to both Gallai’s and Hajós’ Conjectures. In a forthcoming work we apply this technique to verify Conjectures 1 and 2 for partial 3-trees.

**Referências**


